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# The Subclasses of Characterization on $\Pi^*$ -Regular Semigroups

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**Abstract:** In the paper, we define the equivalence relations on  $\Pi^*$ -regular semigroups, to show  $L^*$ ,  $R^*$ ,  $H^*$ ,  $J^*$ -class contains an idempotent with some characterizations.

**Key words:**  $\Pi^*$ -regular semigroup; Completely  $\Pi^*$ -regular semigroup; Subclass; Idempotent

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## 1. INTRODUCTION

An element  $d$  of a semigroup  $S$  is  $\Pi^*$ -regular, if  $S_1$  and  $S_2$  are non-empty regular semigroups, and  $S$  is all non-empty subset of  $S_1 \times S_2$ , for any  $a \in S_1$ ,  $b \in S_2$ ,  $(a, b) = d \in S$ , there exist  $m \in \mathbb{Z}^+$ ,  $(x, y) \in S$ , such that

$$(a, b)^m = (a, b)^m(x, y)(a, b)^m.$$

A semigroup  $S$  is  $\Pi^*$ -regular, if every element of  $S$  is  $\Pi^*$ -regular. A semigroup  $S$  is completely  $\Pi^*$ -regular, if  $S$  is  $\Pi^*$ -regular, for any  $(a, b) \in S$  and element  $(a, b)$  is a regular, there exist  $m \in \mathbb{Z}^+$ ,  $(x, y) \in S$ , such that  $(a, b)^m(x, y) = (x, y)(a, b)^m$  [1].

In this paper, we consider some special case of  $\Pi^*$ -regular semigroups and completely  $\Pi^*$ -regular semigroups.

**Remark** The marks we don't illustrate in this paper please see reference [2].

Now let  $S$  be a  $\Pi^*$ -regular semigroup. Define the equivalence relations  $L^*$ ,  $R^*$ ,  $H^*$ ,  $J^*$  on  $S$  by

$$\begin{aligned} (a, b)L^*(x, y) &\Leftrightarrow S(a, b)^m = S(x, y)^n \\ (a, b)R^*(x, y) &\Leftrightarrow (a, b)^m S = (x, y)^n S \\ H^* &\Leftrightarrow L^* \cap R^* \\ (a, b)J^*(x, y) &\Leftrightarrow S(a, b)^m S = S(x, y)^n S \end{aligned}$$

where  $m, n$  are the smallest positive integers such that  $(a, b)^m$ ,  $(x, y)^n$  are regular, i.e.,  $(a, b)^m$ ,  $(x, y)^n \in S$ . In what follows we will denote by  $L_{(a,b)}^*(R_{(a,b)}^*, H_{(a,b)}^*, J_{(a,b)}^*)$  the  $L^*$ -,  $R^*$ -,  $H^*$ -,  $J^*$ -class containing an element  $(a, b)$  of  $S$ . According to the hypothesis, we can easily draw the following lemma.

**Lemma 1.** Let  $S$  be a  $\Pi^*$ -regular semigroup. Then every  $L^*(R^*)$ -class contains at least one idempotent.

**Lemma 2.** Let  $S$  be a  $\Pi^*$ -regular semigroup. Then each idempotent  $(e, e)$  of  $S$  is a right (left, two-sided) identity for regular elements from  $L_{(e,e)}^*(R_{(e,e)}^*, J_{(e,e)}^*)$ .

**Lemma 3.** In a  $\Pi^*$ -regular semigroup  $S$  every  $H^*$ -class contains at most one idempotent.

**Lemma 4.** Let  $S$  be a  $\Pi^*$ -regular semigroup,  $(a, b) \in S$  and  $p$  be the smallest positive integers such that  $(a, b)^p \in S$ . Then  $(a, b)^p \in L_{(a,b)}^* \cap R_{(a,b)}^* = H_{(a,b)}^*$ .

## 2. MAIN RESULTS

Let  $V$  be the set of all inverse elements of  $S$  [3],  $E$  is the set of all idempotent of  $S$ . Here we get a good result.

**Theorem 1.** Let  $(a, b)$  and  $(x, y)$  be element of a  $\Pi^*$ -regular semigroup  $S$ . Then

$$\begin{aligned} (1) & (a, b)L^*(x, y) \Leftrightarrow \exists (a, b)', (x, y)' \in V, (a, b)'(a, b)^m = (x, y)'(x, y)^n; \\ (2) & (a, b)R^*(x, y) \Leftrightarrow \exists (a, b)', (x, y)' \in V, (a, b)^m(a, b)' = (x, y)^n(x, y)'; \\ (3) & (a, b)H^*(x, y) \Leftrightarrow \exists (a, b)', (x, y)' \in V, (a, b)'(a, b)^m = (x, y)'(x, y)^n, \\ & (a, b)^m(a, b)' = (x, y)^n(x, y)'. \end{aligned}$$

*Proof.* Obviously we only need to prove (3). Let  $(a, b), (x, y) \in H^*$  and  $(a, b)', (x, y)' \in V$ . Then

$$\begin{aligned} (e, e) &= (a, b)'(a, b)^m \in L_{(a,b)}^* \cap E = L_{(x,y)}^* \cap E; \\ (f, f) &= (a, b)^m(a, b)' \in R_{(a,b)}^* \cap E = R_{(x,y)}^* \cap E; \end{aligned}$$

So

$$\begin{aligned} (x, y)^n &= (x, y)^n(e, e) = (f, f)(x, y)^n \\ (f, f) &= (x, y)^n(u, v), (e, e) = (s, t)(x, y)^n ((u, v), (s, t) \in S). \end{aligned}$$

Assume that  $(x, y)' = (e, e)(u, v)(f, f)$  then  $(x, y)' \in V$ . We will find

$$\begin{aligned} (x, y)^n(x, y)'(x, y)^n &= (x, y)^n(e, e)(u, v)(f, f)(x, y)^n \\ &= (x, y)^n(u, v)(x, y)^n = (f, f)(x, y)^n = (x, y)^n; \\ (x, y)'(x, y)^n(x, y) &= (e, e)(u, v)(f, f)(x, y)^n(e, e)(u, v)(f, f) \\ &= (e, e)(u, v)(x, y)^n(u, v)(f, f) = (e, e)(u, v)(f, f) = (x, y)', \end{aligned}$$

Now

$$\begin{aligned}(x, y)^n(x, y)' &= (x, y)^n(e, e)(u, v)(f, f) \\ &= (x, y)^n(u, v)(f, f) = (f, f) = (a, b)^m(a, b)';\end{aligned}$$

$$\begin{aligned}(x, y)'(x, y)^n &= (e, e)(u, v)(f, f)(x, y)^n = (s, t)(x, y)^n(u, v)(f, f)(x, y)^n \\ &= (s, t)(f, f)(x, y)^n = (s, t)(x, y)^n = (e, e) = (a, b)'(a, b)^m.\end{aligned}$$

Conversely, if  $(a, b)'(a, b)^m = (x, y)'(x, y)^n$  and  $(a, b)^m(a, b)' = (x, y)^n(x, y)'$  for some  $(a, b)', (x, y)' \in V$ , then

$$\begin{aligned}(a, b)^m &= (a, b)^m(a, b)'(a, b)^m = (a, b)^m(x, y)'(x, y)^n = (x, y)^n(x, y)'(a, b)^m \\ (x, y)^n &= (x, y)^n(x, y)'(x, y)^n = (x, y)^n(a, b)'(a, b)^m = (a, b)^n(a, b)'(x, y)^n\end{aligned}$$

hence

$$S(a, b)^m = S(x, y)^n, \quad (a, b)^m S = (x, y)^n S$$

whence  $(a, b)H^*(x, y)$ . □

**Theorem 2.** Let  $(e, e)$  be an idempotent of a  $\Pi^*$ -regular semigroup  $S$ . Then  $G_{e,e} \subseteq H_{(e,e)}^*$ , furthermore, if  $(u, v) \in H_{(e,e)}^*$  and  $p$  is the smallest positive integer such that  $(u, v)^p \in S$ , then  $(u, v)^q \in G_{(e,e)}$  for every  $q \geq p$ .

*Proof.* Let  $(a, b) \in G_{(e,e)}$  and let  $(s, t)$  be an inverse element for  $(a, b)$  in  $G_{(e,e)}$  [4]. Since  $(s, t)(a, b) = (e, e) = (a, b)(s, t)$ , we obtain that  $(s, t) \in V$ , so by theorem 1  $(a, b) \in H_{(e,e)}^*$ . Hence  $G_{(e,e)} \subseteq H_{(e,e)}^*$ .

Assume  $(u, v) \in H_{(e,e)}^*$  and let  $p$  be the smallest positive integer such that  $(u, v)^p \in S$ . By theorem 1 (3) there exists  $(u, v)' \in V$  such that  $(u, v)'(u, v)^p = (e, e) = (u, v)^p(u, v)'$ . So  $(u, v)^p$  is completely regular, i.e.,  $(u, v)^p \in G_{(f,f)}$ . It is easy to show that  $(e, e) = (f, f)$ . Therefore  $(u, v)^q \in G_{(e,e)}$ ,  $q \geq p$ . □

By theorem 1 and theorem 2, we obtain the following case.

**Proposition.**  $S$  is a  $\Pi^*$ -regular semigroup if and only if  $S$  is completely regular semigroup and  $H^*$ -class contains an idempotent.

**Lemma 5.** Let  $S$  be a  $\Pi^*$ -regular semigroup. Then

- (1) every  $J^*$ -class contains at least one idempotent;
- (2)  $G_{(e,e)} \subseteq H_{(e,e)} \subseteq J_{(e,e)}$  for every  $e \in E$ .

**Lemma 6.** Let  $S$  be a  $\Pi^*$ -regular semigroup. Then (for some  $(e, e), (f, f) \in E$ )

$$J_{(e,e)}^* = J_{(f,f)}^*, (e, e)(f, f) = (f, f)(e, e) = (f, f) \Rightarrow (e, e) = (f, f).$$

*Proof.* It follows from that  $S(e, e)S = S(f, f)S$  that

$$(e, e) = (u, v)(f, f)(s, t) \quad ((u, v), (s, t) \in S).$$

Suppose that

$$(a, b) = (e, e)(u, v)(e, e),$$

then

$$\begin{aligned}(a, b) &= (e, e)(u, v)(e, e) \\ &= (e, e)(u, v)(e, e)(f, f)(s, t)(e, e)(u, v)(e, e) \\ &= (a, b)(f, f)(s, t)(a, b).\end{aligned}$$

By the hypothesis that there exists  $(a, b)' \in S$ , such that

$$(a, b) = (a, b)(a, b)'(a, b), \quad (a, b)'(a, b) = (a, b)(a, b)'.$$

Now we assume  $(x, y) = (e, e)(s, t)(e, e)$ , then

$$\begin{aligned} (a, b)(f, f)(x, y) &= (e, e)(u, v)(e, e)(f, f)(e, e)(s, t)(e, e) \\ &= (e, e)(u, v)(f, f)(s, t)(e, e) = (e, e) \end{aligned}$$

Therefore,

$$\begin{aligned} (e, e) &= (a, b)(f, f)(x, y) = (a, b)(a, b)'(a, b)(f, f)(x, y) \\ &= (a, b)(a, b)'(e, e) = (a, b)'(a, b)(e, e) = (a, b)'(a, b). \end{aligned}$$

Hence,

$$\begin{aligned} (e, e) &= (a, b)'(a, b) = (a, b)'(e, e)(a, b) \\ &= (a, b)'(a, b)(f, f)(x, y)(a, b) = (e, e)(f, f)(x, y)(a, b)(f, f)(x, y)(a, b). \end{aligned}$$

Thus

$$(e, e) = (f, f)(e, e)$$

and whence

$$(e, e) = (f, f).$$

□

**Lemma 7.** Let  $S$  be a  $\Pi^*$ -regular semigroup. Then for some  $(u, v) \in S$ ,  $(e, e) \in E$ ,

$$J_{(e,e)}^* = J_{(e,e)(u,v)(e,e)}^* \Rightarrow (e, e)(u, v)(e, e) \in G_{(e,e)}.$$

*Proof.* This is obvious. Here we don't prove it. □

**Theorem 3.** Let  $S$  be a  $\Pi^*$ -regular semigroup. Then (for some  $(a, b)$ ,  $(x, y) \in S$ ),

$$J_{(a,b)(x,y)}^* = J_{(x,y)(a,b)}^*.$$

*Proof.* Let  $p$  and  $q$  be the smallest positive integers such that

$$((a, b)(x, y))^p, ((x, y)(a, b))^q \in S.$$

Then

$$\begin{aligned} ((a, b)(x, y))^p &\in G_{(e,e)} \subseteq J_{(e,e)}^*, \\ ((x, y)(a, b))^q &\in G_{(f,f)} \subseteq J_{(f,f)}^*. \end{aligned}$$

Whence by [5] we obtain

$$\begin{aligned} ((a, b)(x, y))^{p+m} &\in G_{(e,e)} \subseteq J_{(e,e)}, \\ ((x, y)(a, b))^{q+n} &\in G_{(f,f)} \subseteq J_{(f,f)}^*. \end{aligned}$$

For every  $p, q \geq 0$ , so

$$\begin{aligned} ((a, b)(x, y))^m J^*((a, b)(x, y))^{p+m}, \\ ((x, y)(a, b))^n J^*((x, y)(a, b))^{q+n}. \end{aligned}$$

And for  $k = \max(m, n)$ , we have

$$\begin{aligned} S((a, b)(x, y))^m S &= S((a, b)(x, y))^{k+1} S, \\ &= S(a, b)((x, y)(a, b))^k (x, y) S \subseteq S((x, y)(a, b))^k S \subseteq S((x, y)(a, b))^n S. \end{aligned}$$

Similarly,

$$S((x, y)(a, b))^n S \subseteq S((a, b)(x, y))^m S.$$

Thus,

$$S((a, b)(x, y))^m S = S((x, y)(a, b))^n S.$$

That is

$$J_{(a,b)(x,y)}^* = J_{(x,y)(a,b)}^*.$$

□

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